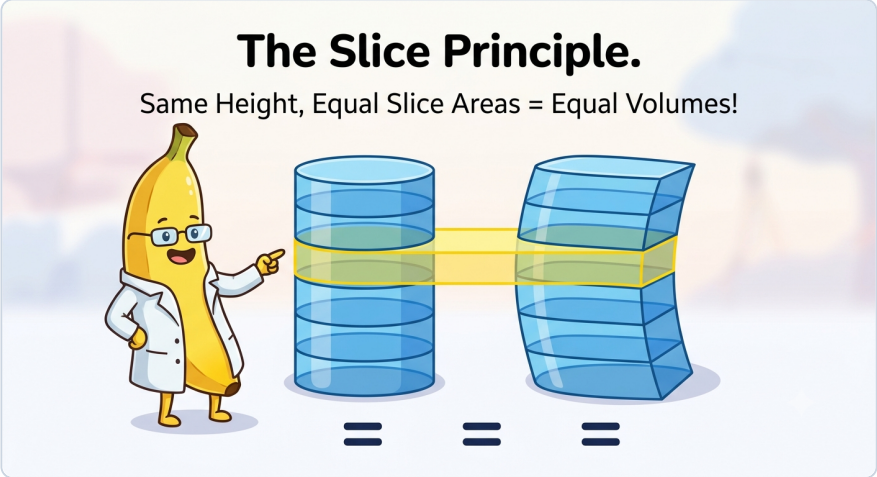
Visual Proof: Volume of a Sphere

Guided by Nano Banana • Archimedes' Method via Cavalieri's Principle

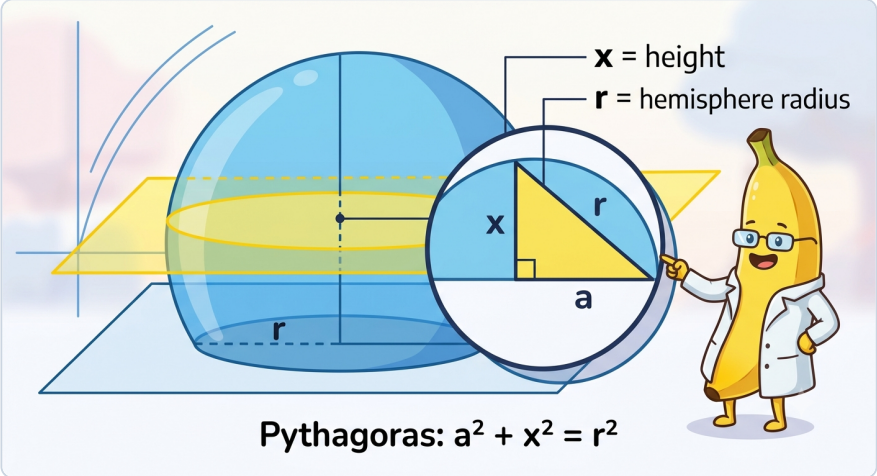
Target Formula: $V = \frac{4}{3} \pi r^3$

Step 1. The Slice Principle (Cavalieri's Principle)



Nano Banana establishes our foundational principle: If two solids have the same total height, and at every corresponding height their horizontal cross-sectional slice areas are equal, then their total volumes must also be equal.

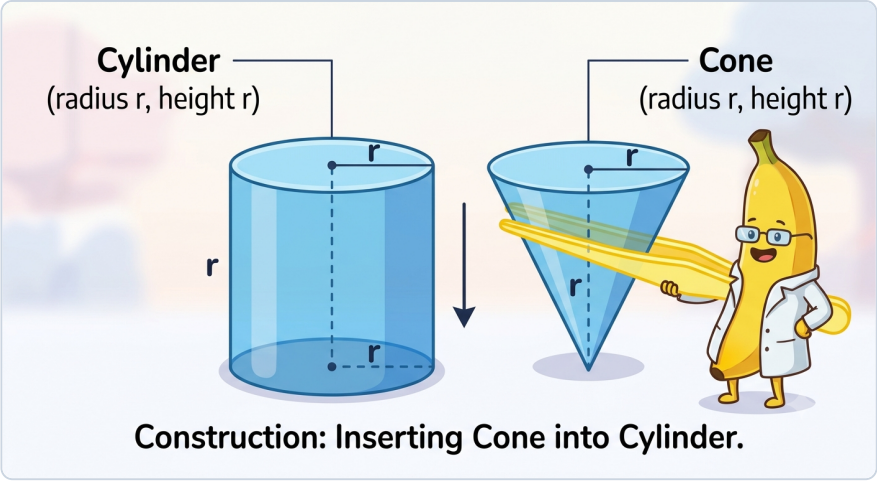
Step 2. Analyzing the Hemisphere Cross-Section



Consider a hemisphere of radius r and height r. Making a horizontal cut at height x (where $0 \le x \le r$) yields a circular slice of radius a.
A vertical cross-section through the center reveals a right-angled triangle formed by height x, slice radius a, and sphere radius r.

> By Pythagoras: $a^2 + x^2 = r^2 \Rightarrow a^2 = r^2 - x^2$
> **Slice Area of Hemisphere at height x:** $\pi a^2 = \pi(r^2 - x^2)$

Step 3. Building the Comparison Solid

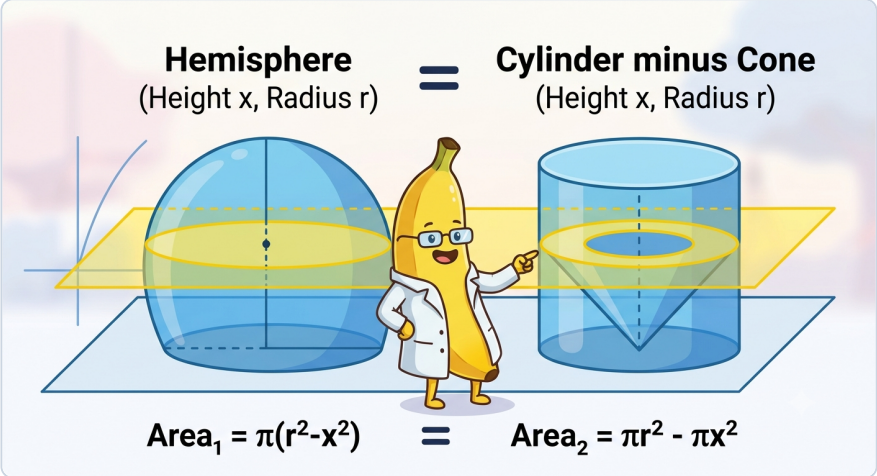


To use the Slice Principle, Archimedes created a comparison solid with height r and base radius r:

- Start with a solid cylinder of radius r and height r.
- Remove an inverted cone of radius r and height r (apex at the bottom center, base at the top).

The comparison solid consists of the cylinder volume with the cone removed.

Step 4. Side-by-Side Slice Comparison

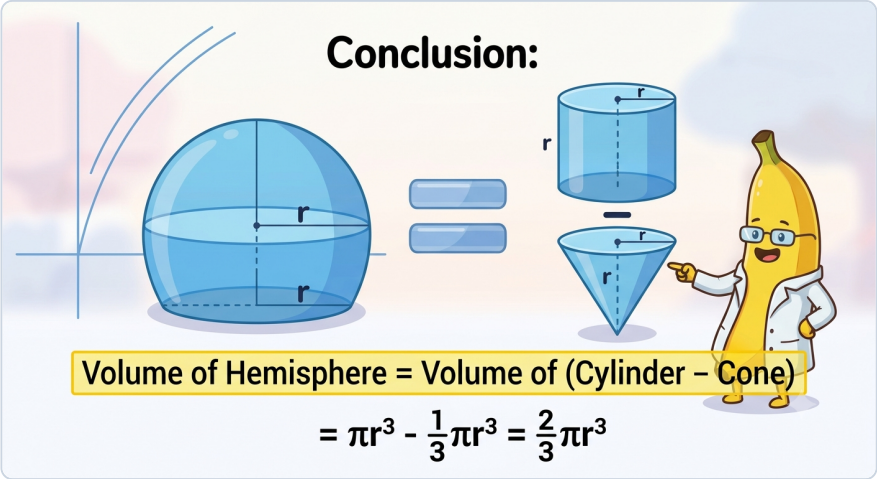


Cutting both solids horizontally at height x:

- **Cylinder slice area:** πr^2
- **Cone slice area removed:** By similar triangles, the cone's radius at height x is x, giving area πx^2.
- **Remaining comparison slice area (Ring/Washer):** $\pi r^2 - \pi x^2 = \pi(r^2 - x^2)$.

> Both slices have identical cross-sectional area $\pi(r^2 - x^2)$ at every height x!

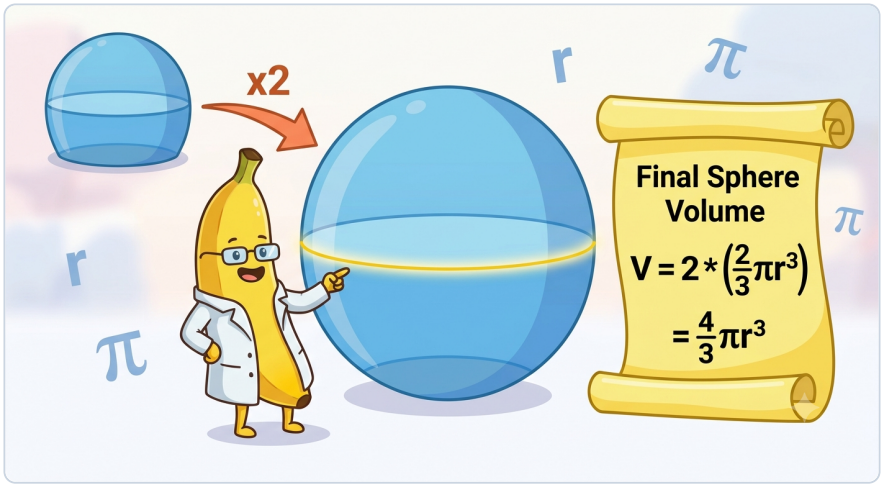
Step 5. Hemisphere Volume Calculation



Since corresponding cross-sectional areas are identical at all heights, their volumes must be equal:

> $V_{hemisphere} = V_{cylinder} - V_{cone}$
> $V_{hemisphere} = (\pi r^2 \cdot r) - (\frac{1}{3}\pi r^2 \cdot r) = \pi r^3 - \frac{1}{3}\pi r^3 = \frac{2}{3}\pi r^3$

Step 6. Doubling to Get Full Sphere Volume



A full sphere is composed of two identical hemispheres stacked base-to-base.

$$V_{sphere} = 2 \cdot V_{hemisphere} = 2 \cdot (^2/_3\,\pi r^3) = {}^4/_3\,\pi r^3$$

SUMMARY FOR STUDENTS

1. A horizontal slice of a hemisphere at height x has area $\pi(r^2 - x^2)$ via Pythagoras.
2. A cylinder minus an inverted cone has slice area $\pi r^2 - \pi x^2 = \pi(r^2 - x^2)$ at height x.
3. By Cavalieri's Principle, equal slice areas at every height imply equal total volume.
4. $V_{hemisphere} = V_{cylinder} - V_{cone} = \pi r^3 - {}^1/_3\pi r^3 = {}^2/_3\pi r^3$.
5. Doubling gives the volume of a sphere: $V_{sphere} = {}^4/_3\pi r^3$.